P.E.S. COLLEGE OF ENGINEERING

(AN AUTONOMOUS INSTITUTE)

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Regular Winter Examination – 2025

Course: F.Y.B. Tech. **Branch : ALL** **Semester : I**

Subject Code & Name: BTPES101T Engineering Mathematics-I

Max Marks: 60 **Date:28/01/2026** **Duration: 3 Hr.**

Instructions to the Students:

1. *All the questions are compulsory.*
2. *The level of question/expected answer as per OBE or the Course Outcome (CO) on which the question is based is mentioned in ( ) in front of the question.*
3. *Use of non-programmable scientific calculators is allowed.*
4. *Assume suitable data wherever necessary and mention it clearly.*

		(Level/CO)	Marks
Q. 1	**Solve Any six of the following.**		**6x2=12**
A)	*Reduce to the normal form and find the rank of the matrix* $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$	CO1	2
B)	*Find* A^{-1} *by Cayley – Hamilton theorem for the matrix* $A = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$	CO1	2
C)	*Define linear dependence and independence of vectors.*	CO1	2
D)	$If\ u = x - y\ and\ x = e^t, y = t^2, find\ \frac{du}{dt}$	CO2	2
E)	$If\ u = \frac{x^3}{y^2}, then\ find\ the\ value\ of\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$	CO2	2
F)	$If\ u = logxy, then\ prove\ that\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x+y}{xy}$	CO2	2
G)	*Find the solution of exact differential equation* $(1 + e^x)dx + ydy = 0$	CO3	2
H)	*Find the integrating factor of the linear differential equation* $\frac{dy}{dx} + \frac{2y}{x} = x^2$	CO3	2
I)	*Obtain the orthogonal trajectory of the family of curves* $ay = x$	CO3	2
Q.2	**Solve Any Two of the following.**		12
A)	*Test the consistency and solve* $3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y + z = 4$	CO1	6
B)	*Find the eigenvalues & eigenvectors for the matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$	CO1	6

C)	*Trace the curve* $3ay^2 = x(x-a)^2$	CO4	6
Q. 3	**Solve Any Two of the following.**		12
A)	*If* $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *prove that* $\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}$	CO2	6
B)	*If* $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, *prove that* $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$	CO2	6
C)	*Trace the curve* $r = a\cos 2\theta$ *with full justification*	CO4	6
Q.4	**Solve Any Two of the following.**		12
A)	*If* $x = u(1-v), y = uv$, *prove that* $JJ' = 1$	CO3	6
B)	*Divide 24 into three parts such that the continued product of the first, square of second and cube of third may be maximum.*	CO3	6
C)	*Expand* $f(x,y) = e^x \sin y$ *in powers of* x & y *as far as the terms of third degree*	CO3	6
Q. 5	**Solve Any Two of the following.**		12
A)	*Solve* $ydx - xdy + \log x dx = 0$	CO5	6
B)	*A constant electromotive force E volts is applied to a circuit containinga constant resistance R ohm in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theorotical maximum in* $\frac{L \log 2}{R}$ *sec.*	CO5	6
C)	*Find the length of the curve* $x = a[\cos t + t\sin t]$, $y = a[\sin t - t\cos t]$ *from* $t = 0$ *to* $t = \frac{\pi}{2}$	CO4	6
	*** **End** ***		

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Subject :- Engg. maths - I FY (All) 28/01/2026
(BTPES101T)

Q.1] Solve any SIX of the following. [12 M]

Q.1] A) Reduce to the normal form and find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ [2 M]

Answer → Given $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

step-1 $R_3 + R_1$

$A \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

step-2 $(-1)R_2$, $-\frac{1}{4}R_3$

$A \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

step-3 $R_1 + 4R_3$, $R_2 + R_3$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I_3]$

$\therefore \rho(A) = 3$.

Q.1] B) Find A^{-1} by Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$ [2 M]

Answer → The characteristic eqn is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 5-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$ $\therefore (5-\lambda)(2-\lambda) - 0 = 0$

$\therefore \lambda^2 - 7\lambda + 10 = 0$

by Cayley Hamilton theorem $A^2 - 7A + 10I = 0$.

multiply by A^{-1} $\therefore A - 7I + 10A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{10}[7I - A]$

Now, $A^2 = A \cdot A = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25+0 & 0+0 \\ 5+2 & 0+4 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 25 & 0 \\ 7 & 4 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{10}\left[7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}\right] = \frac{1}{10}\begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

Q.1] c) Define linear dependence and independence of vectors. [2m]

Answer → <u>Linear Dependence</u> → A set of r vectors $X_1, X_2, \cdots, X_r$ is said to be linearly dependent if there exists r scalars (numbers) $k_1, k_2, \cdots, k_r$ not all zero, such that

$$k_1 X_1 + k_2 X_2 + \cdots + k_r X_r = 0$$

<u>Linear Independence</u> → A set of r vectors $X_1, X_2, \cdots, X_r$ is said to be linearly independent if there exists r scalars (numbers) $k_1, k_2, \cdots, k_r$, such that if

$$k_1 X_1 + k_2 X_2 + \cdots + k_r X_r = 0$$

then $k_1 = k_2 = \cdots = k_r = 0$.

Q.1] d) If ~~$u = \frac{x^3}{y^2}$, then find the~~

If $u = x - y$ & $x = e^t$, $y = t^2$, find $\frac{du}{dt}$. [2m]

Answer → $\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$= 1 \cdot e^t + (-1) 2t$

$\therefore \frac{du}{dt} = e^t - 2t$

Q.1] E) If $u = \frac{x^3}{y^2}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. [2M]

Answer → Given $u = \frac{x^3}{y^2}$

put $x = xt$, $y = yt$ $\Rightarrow u = \frac{x^3 t^3}{y^2 t^2} = t^1 \cdot \frac{x^3}{y^2}$

∴ u is homogeneous funⁿ in x & y of degree $n = 1$.

∴ by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \cdot u = u = \frac{x^3}{y^2}$.

OR $\frac{\partial u}{\partial x} = \frac{3x^2}{y^2}$ $\therefore x\frac{\partial u}{\partial x} = \frac{3x^3}{y^2}$

$\frac{\partial u}{\partial y} = \frac{-2x^3}{y^3}$ $\therefore y\frac{\partial u}{\partial y} = \frac{-2x^3}{y^2}$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3}{y^2}$.

Q.1] F) If $u = \log xy$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x+y}{xy}$. [2M]

Answer → $\frac{\partial u}{\partial x} = \frac{1}{xy} \cdot y = \frac{y}{xy}$

$\frac{\partial u}{\partial y} = \frac{1}{xy} \cdot x = \frac{x}{xy}$.

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x+y}{xy}$

Q.1] G) Find the solution of exact differential equation
$(1+e^y)\,dx + y\,dy = 0$. [2M]

Answer → The solution of exact differential eqⁿ is

$$\int_{y\text{ const.}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = C$$

$$\int (1+e^y)\,dx + \int y\,dy = C$$

$$\therefore\ x + e^y + \frac{y^2}{2} = C.$$

Q.1] H) Find the integrating factor of the linear differential eqⁿ $\frac{dy}{dx} + \frac{2y}{x} = x^2$. [2M]

Answer → Compare with $\frac{dy}{dx} + Py = Q$

$\therefore\ P = \frac{2}{x}$, $Q = x^2$

$$I.F. = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$$\therefore\ \boxed{I.F. = x^2}$$

Q.1] I) Obtain the orthogonal trajectory of $ay = x$. [2M]

Answer → Given $ay = x \rightarrow (1)$

diff. w.r.t. x

$a\frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{1}{a}$

from (1) $a = \frac{x}{y}$

$\therefore\ \frac{1}{a} = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$

put $dy/dx = -dx/dy$

$\therefore -dx/dy = y/x \Rightarrow -x\,dx = y\,dy$

$\therefore\ x\,dx + y\,dy = 0$

integrating $\Rightarrow x^2/2 + y^2/2 = a^2/2$

$\therefore\ \boxed{x^2 + y^2 = a^2}$

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Q.2] Solve any TWO of the following. [12 M]

Q.2] A> Test the consistency & solve

$3x+y+2z=3, \quad 2x-3y-z=-3, \quad x+2y+z=4.$ [6M].

Answer → The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

step 1 $R_1 \longleftrightarrow R_3$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

step-2 $R_2 - 2R_1, \; R_3 - 3R_1$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

step-3 $-\frac{1}{7}R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 3/7 & 11/7 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

step-4 $R_3 + 5R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 3/7 & 11/7 \\ 0 & 0 & 8/7 & -8/7 \end{array}\right]$$

$\therefore \rho(A) = \rho(A|B) = 3 =$ no. of unknowns.

$\therefore$ system is consistent & has unique solution.

rewriting the system

$x+2y+z=4$

$y + \frac{3z}{7} = \frac{11}{7}$

$\frac{8z}{7} = -\frac{8}{7} \Rightarrow z=-1$

$y - \frac{3}{7} = \frac{11}{7} \Rightarrow y = \frac{14}{7} \Rightarrow y=2$

$x+4-1=4 \Rightarrow x=1$

$\therefore$ solution is $x=1, y=2, z=-1$.

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Q.2] B) Find the eigenvalues & eigenvectors for the matrix $A=\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$ [6m]

Answer → The characteristic eqn is $|A-\lambda I|=0$.

$$\begin{vmatrix}3-\lambda&1&4\\0&2-\lambda&6\\0&0&5-\lambda\end{vmatrix}=0$$

$\therefore (3-\lambda)[(2-\lambda)(5-\lambda)-0]-1[0-0]+4[0-0]=0$

$\therefore (3-\lambda)(2-\lambda)(5-\lambda)=0$

$\therefore \lambda = 2,3,5$ are the eigenvalues.

To find the eigenvectors →

The matrix equation is $[A-\lambda I]X=0$.

$$\therefore \begin{bmatrix}3-\lambda&1&4\\0&2-\lambda&6\\0&0&5-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow (1)$$

(1) For $\lambda=2$ put $\lambda=2$ in (1)

$$\begin{bmatrix}1&1&4\\0&0&6\\0&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore x+y+4z=0$

$0x+0y+6z=0$

$\therefore$ by Cramer's rule

$$\frac{x}{\begin{vmatrix}1&4\\0&6\end{vmatrix}}=\frac{-y}{\begin{vmatrix}1&4\\0&6\end{vmatrix}}=\frac{z}{\begin{vmatrix}0&1\\0&0\end{vmatrix}} \Rightarrow \frac{x}{6-0}=\frac{-y}{6-0}=\frac{z}{0-0}$$

$$\therefore \frac{x}{6}=\frac{y}{-6}=\frac{z}{0} \Rightarrow \frac{x}{1}=\frac{y}{-1}=\frac{z}{0}$$

~~$\therefore$ eigenvalu~~ $\therefore$ eigenvector for $\lambda=2$ is $\begin{bmatrix}1\\-1\\0\end{bmatrix}$.

② For $\lambda = 3 \rightarrow$ put $\lambda = 3$ in ①

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 0x + y + 4z = 0$

$0x - y + 6z = 0$

by Cramer's rule

$$\frac{x}{\begin{vmatrix} 1 & 4 \\ -1 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix}} \Rightarrow \frac{x}{6+4} = \frac{-y}{0-0} = \frac{z}{0-0}$$

$$\therefore \frac{x}{10} = \frac{y}{0} = \frac{z}{0} \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

$\therefore$ eigenvector for $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

③ For $\lambda = 5 \longrightarrow$ put $\lambda = 5$ in ①

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 4z = 0$

$0x - 3y + 6z = 0$

$\therefore$ by Cramer's rule

$$\frac{x}{\begin{vmatrix} 1 & 4 \\ -3 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix}} \Rightarrow \frac{x}{6+12} = \frac{-y}{-12-0} = \frac{z}{6-0}$$

$$\therefore \frac{x}{18} = \frac{y}{12} = \frac{z}{6} \Rightarrow \frac{x}{3} = \frac{y}{2} = \frac{z}{1}$$

$\therefore$ eigenvector for $\lambda = 5$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ eigenvalues are 3, 2, 5 &

eigenvector are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Q.2] c) Trace the curve $3ay^2 = x(x-a)^2$ [6 M]

Answer →

Given $3ay^2 = x(x-a)^2 \longrightarrow ①$

$\therefore 3ay^2 = x(x^2 - 2ax + a^2)$

$\therefore 3ay^2 - x^3 + 2ax^2 - a^2x = 0$.

① Symmetry → As all the powers of y are even ∴ Curve is symmetrical about X-axis.

② Origin → The eq^n of curve does not contain any constant term. ∴ curve passes through origin. To find the nature of curve at origin, find the tangents to the curve at origin by equating lowest degree term to zero.

$\therefore -a^2x = 0 \Rightarrow x = 0$ ∴ Y-axis is a single tangent to the curve at origin.

∴ Curve behaves in eliptical manner at origin.

③ Point of intersection with X-axis → put $y=0$ in ①

$\therefore 0 = x(x-a)^2 \Rightarrow x = 0, x = a$.

∴ Points of intersection are $(0,0)$, $(a,0)$.

To find nature of curve at $(a,0)$ → shift origin to $(a,0)$ & then find the tangents to the curve at shifted origin by equating lowest degree terms to zero.

$\therefore (0,0) \rightarrow (a,0) \therefore x \rightarrow x+a$ & $y \rightarrow y+0$ put in ①

$\therefore 3ay^2 = (x+a)(x+a-a)^2 \Rightarrow 3ay^2 - x^3 - ax^2 = 0$

$\therefore 3ay^2 - ax^2 = 0$

$\Rightarrow y^2 = \frac{x^2}{3} \Rightarrow y = \frac{x}{\sqrt{3}}, y = \frac{-x}{\sqrt{3}}$ are two different tangents to the curve at shifted origin $(a,0)$

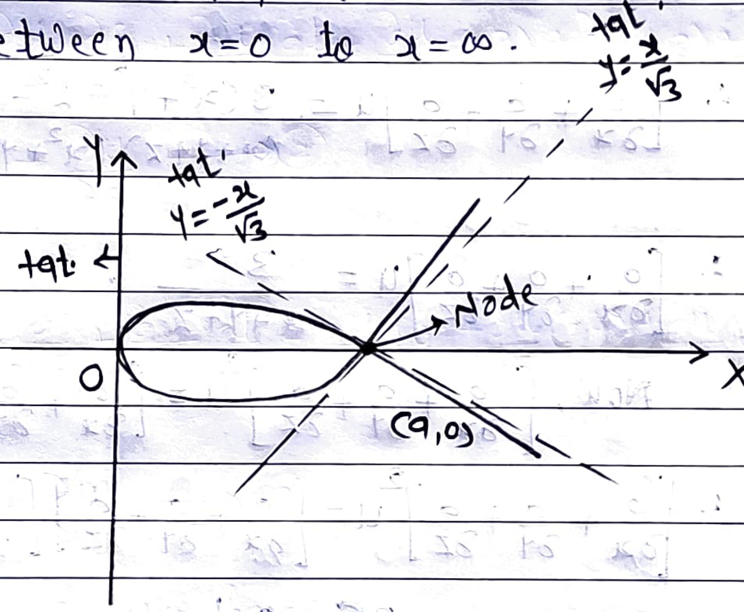
∴ $(a,0)$ is a node point.

④ Asymptote parallel to y-axis →
equate coefficient of highest degree term to zero.
$\therefore 3a = 0$
this does not represent any equation of straight line.
$\therefore$ No asymptote.

⑤ Region of existence –
$\therefore$
$$y^2 = \frac{x(x-a)^2}{3a}$$

If $x<0$ then y^2 is negative & y will be imaginary.
$\therefore$ there is no curve when $x<0$.
$\therefore$ curve lies between $x=0$ to $x=\infty$.

⑥ Diagram →



Q.3] Solve any TWO of the following. [12M].

Q.3]A) If $u = \log(x^3+y^3+z^3-3xyz)$, prove that
$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}$$ [6M]

Answer → Given: $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2-3yz) = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3y^2-3xz) = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3z^2-3xy) = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$\therefore \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]u = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\therefore \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]u = \frac{3}{x+y+z}$$

Now, $\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]u$

$$\therefore \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]\left[\frac{3}{x+y+z}\right]$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

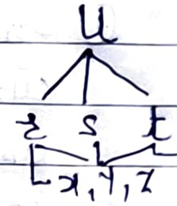
$$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{(-1)}{(x+y+z)^2} + \frac{(-1)}{(x+y+z)^2}\right]$$

$$\therefore \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}$$

Q.3] B) If $u = f(2x-3y, 3y-4z, 4z-2x)$, prove that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$ [6M].

Answer →



put $\xi = 2x-3y,\ s = 3y-4z,\ t = 4z-2x$

$\therefore u = f(\xi, s, t)$.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial u}{\partial \xi}\cdot 2 + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}(-2)$$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial t} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial u}{\partial \xi}(-3) + \frac{\partial u}{\partial s}(3) + \frac{\partial u}{\partial t}(0)$$

$$\therefore \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial s} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial u}{\partial \xi}(0) + \frac{\partial u}{\partial s}(-4) + \frac{\partial u}{\partial t}(4)$$

$$\therefore \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \longrightarrow (3)$$

adding (1), (2) & (3), we have

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

Q.3] c) Trace the curve $r = a\cos 2\theta$ with full justification. [6M].

Answer →

① Symmetry – If θ is replaced by $-\theta$, the equation of curve remains unchanged. ∴ curve is symmetrical about initial line.

② Pole – when $\theta = \pm \pi/2$ then $r = 0$
∴ curve passes through pole (origin).
put $r = 0 \Rightarrow 0 = a\cos 2\theta \Rightarrow 2\theta = \cos^{-1}(0) = \pm \pi/2$
∴ $\theta = \pm \frac{\pi}{4}$ are the tangents to the curve at pole.

③ $\tan\phi = r\frac{d\theta}{dr} = \frac{a\cos 2\theta}{-2a\sin 2\theta} = -\frac{1}{2}\cot 2\theta = \frac{1}{2}\tan\left(\frac{\pi}{2} + 2\theta\right)$.

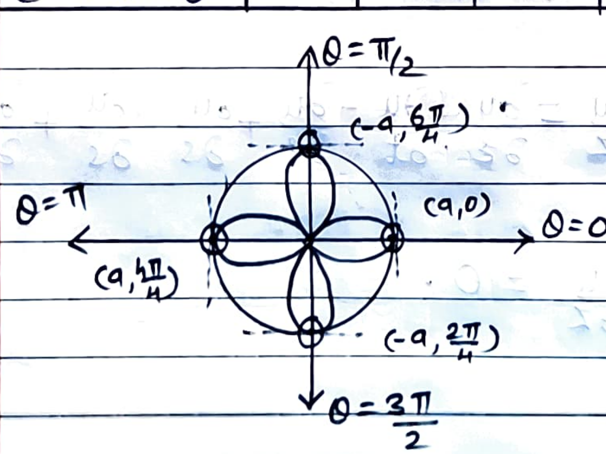
when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ then $\phi = 0$.

when $\theta = 0, \frac{2\pi}{4}, \frac{4\pi}{4}, \frac{6\pi}{4}, \ldots$ then $\phi = \frac{\pi}{2}$.

④ $-1 \leq \cos 2\theta \leq 1 \Rightarrow -a \leq a\cos 2\theta \leq a \Rightarrow -a \leq r \leq a$
∴ entire curve lies within the circle $r = a$.

⑤ Table

θ	0	$\pi/4$	$2\pi/4$	$3\pi/4$	$4\pi/4$	$5\pi/4$	$6\pi/4$
$r = a\cos 2\theta$	a	0	-a	0	a	0	-a



Q.4] Solve any TWO of following [12m]

Q.4] A) If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. [6m]

Answer →

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv$$

$$\therefore \boxed{J = u}$$

Now, $$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \longrightarrow (1)$$

Given $x = u(1-v) = u - uv$ & $y = uv$

$\therefore x = u - y \Rightarrow u = x + y \quad \therefore \frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 1$

Now, $y = uv \quad \therefore v = \frac{y}{u} \quad \therefore v = \frac{y}{x+y}$

$$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}, \quad \frac{\partial v}{\partial y} = \frac{x+y-y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

put in (1)

$$\therefore J' = \begin{vmatrix} 1 & 1 \\ -y/(x+y)^2 & x/(x+y)^2 \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

$$\therefore \boxed{J' = \frac{1}{u}}$$

$$\therefore J \cdot J' = u \cdot \frac{1}{u} = 1$$

$$\therefore JJ' = 1.$$

Q. 4] B) Divide 24 into three parts such that the continued product of the first, square of second & cube of third may be maximum. [6 M].

Answer → Let x, y, z be the numbers.

Let $f(x,y,z) = xy^2z^3$ → ①

& $\phi(x,y,z) = x+y+z = 24$ → ②

using Lagrange's method of undetermined multipliers

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow y^2z^3 + \lambda = 0 \rightarrow ③$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2xyz^3 + \lambda = 0 \rightarrow ④$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 3xy^2z^2 + \lambda = 0 \rightarrow ⑤$$

multiply eq^n ③ by x, ④ by y & ⑤ by z, we have

$$xy^2z^3 + \lambda x + 2xy^2z^3 + \lambda y + 3xy^2z^3 + \lambda z = 0$$

$\therefore\ 6xy^2z^3 + \lambda(x+y+z) = 0$

$\therefore\ 6xy^2z^3 + 24\lambda = 0$ ($\because$ from ②)

$\therefore\ \lambda = -\dfrac{xy^2z^3}{4}$

put in ③, ④ & ⑤, we have

$$y^2z^3 - \frac{xy^2z^3}{4} = 0 \Rightarrow \frac{xy^2z^3}{4} = y^2z^3 \Rightarrow x = 4$$

$$2xyz^3 - \frac{xy^2z^3}{4} = 0 \Rightarrow \frac{xy^2z^3}{4} = 2xyz^3 \Rightarrow y = 8$$

$$3xy^2z^2 - \frac{xy^2z^3}{4} = 0 \Rightarrow \frac{xy^2z^3}{4} = 3xyz^2 \Rightarrow z = 12$$

$\therefore$ Max. $f(x,y,z) = xy^2z^3 = 4 \times 8^2 \times (12)^3 = 4{,}42{,}368$.

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Q.4] c) Expand $f(x,y) = e^x \sin y$ in powers of x & y as far as the terms of third degree. [6m]

Answer → The Maclaurin's theorem is

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy} + y^2 f_{yy}]$$
$$+ \frac{1}{3!}[x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}] + \cdots \longrightarrow (1)$$

$f(x,y) = e^x \sin y$, $f(0,0) = e^0 \sin 0 = 0$

$f_x = e^x \sin y$, $f_x(0,0) = e^0 \sin 0 = 0$

$f_y = e^x \cos y$, $f_y(0,0) = e^0 \cos 0 = 1$

$f_{xx} = e^x \sin y$, $f_{xx}(0,0) = e^0 \sin 0 = 0$

$f_{xy} = e^x \cos y$, $f_{xy}(0,0) = e^0 \cos 0 = 1$

$f_{yy} = -e^x \sin y$, $f_{yy}(0,0) = -e^0 \sin 0 = 0$

$f_{xxx} = e^x \sin y$, $f_{xxx}(0,0) = e^0 \sin 0 = 0$

$f_{xxy} = e^x \cos y$, $f_{xxy}(0,0) = e^0 \cos 0 = 1$

$f_{xyy} = -e^x \sin y$, $f_{xyy}(0,0) = -e^0 \sin 0 = 0$

$f_{yyy} = -e^x \cos y$, $f_{yyy}(0,0) = -e^0 \cos 0 = -1$

put in (1)

$$\therefore e^x \sin y = 0 + [x \cdot 0 + y \cdot 1] + \frac{1}{2!}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0]$$
$$+ \frac{1}{3!}[x^3 \cdot 0 + 3x^2y \cdot 1 + 3xy^2 \cdot 0 + y^3(-1)] + \cdots$$

$\therefore$

$$e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \cdots$$

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Q.5] Solve any TWO of the following. [12M]

Q.5]A) Solve $y\,dx - x\,dy + \log x\,dx = 0$. [6M]

Answer → Given $(y+\log x)dx - x\,dy = 0$

compare with $M\,dx + N\,dy = 0 \Rightarrow M = y + \log x,\ N = -x$.

$\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -1 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ given differential equation is not exact.

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-(-1)}{-x} = \frac{-2}{x} = f(x)$

$I.F. = e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiply given eq^n by $1/x^2$

$\therefore \left[\frac{y}{x^2} + \frac{\log x}{x^2}\right]dx - \frac{1}{x}dy = 0 \longrightarrow ①$

compare with $M_1 dx + N_1 dy = 0 \Rightarrow M_1 = \frac{y}{x^2} + \frac{\log x}{x^2},\ N_1 = -\frac{1}{x}$

$\frac{\partial M_1}{\partial y} = \frac{1}{x^2},\ \frac{\partial N_1}{\partial x} = \frac{1}{x^2} \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

∴ eq^n ① is exact.

∴ its solution is

$\int_{y\ const.} M_1\,dx + \int(\text{terms of } N_1 \text{ free from } x)\,dy = C$

$\therefore \int\left[\frac{y}{x^2} + \frac{\log x}{x^2}\right]dx + \int 0\,dy = C_1$

$\therefore -\frac{y}{x} - \frac{1}{x}(1+\log x) = C_1$

$\therefore \frac{1}{x}\left[y + 1 + \log x\right] = C$.

Q.5] B) A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohm in series & a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $\frac{L \log 2}{R}$ seconds. [6m]

Answer → The differential eqⁿ of L-R circuit is

$$Ri + L\frac{di}{dt} = E \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

this is linear in i. $P = R/L$, $Q = E/L$

$$I.F. = e^{\int P\,dt} = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

& its solution is $i(I.F.) = \int Q(I.F.)\,dt + C$

∴ ⓐ

$$i e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\,dt + C = \frac{E}{L}\frac{e^{Rt/L}}{R/L} + C$$

$$\therefore i e^{Rt/L} = \frac{E}{R} e^{Rt/L} + C \Rightarrow i = \frac{E}{R} + C e^{-Rt/L}$$

initially at $t = 0$, $i = 0$

$$\therefore 0 = \frac{E}{R} + Ce^{0} \Rightarrow C = -\frac{E}{R}$$

$$\therefore i = \frac{E}{R}\left[1 - e^{-Rt/L}\right] \longrightarrow (1)$$

this is current at any time t.

as $t \to \infty$, current i will be maximum.

∴ put $i = i_{max}$ & $t \to \infty$ in (1)

$$i_{max} = \frac{E}{R}\left[1 - e^{-\infty}\right] = \frac{E}{R}[1 - 0] = \frac{E}{R}$$

$$\frac{i_{max}}{2} = \frac{E}{2R}$$

put in (1), we have

$$\frac{E}{2R} = \frac{E}{R}\left[1 - e^{-Rt/L}\right]$$

$$\therefore \frac{1}{2} = 1 - e^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2}$$

$$\therefore \log e^{-Rt/L} = \log(1/2) \Rightarrow -\frac{Rt}{L}\log e = \log 2^{-1}$$

$$\therefore -\frac{Rt}{L} = -\log 2 \Rightarrow t = \frac{L}{R}\log 2 \text{ seconds}$$

Q.5] c) Find the length of the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ from $t = 0$ to $t = \pi/2$. [6M]

Answer → The length of the curve in parametric form is given by

$$S = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \rightarrow (1)$$

$$\frac{dx}{dt} = a\left[-\sin t + t\cos t + \sin t\right] = at\cos t$$

$$\frac{dy}{dt} = a\left[\cos t + t\sin t - \cos t\right] = at\sin t$$

$$\therefore S = \int_0^{\pi/2} \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}\, dt = \int_0^{\pi/2} \sqrt{a^2t^2(\cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^{\pi/2} at\, dt = a\left[\frac{t^2}{2}\right]_0^{\pi/2} = \frac{a}{2}\left[\frac{\pi^2}{4} - 0\right]$$

$$\therefore S = \frac{a\pi^2}{8}.$$